

The Moving Frame, Curvature, and Torsion

Calculus III · Intrinsic Geometry of Space Curves

Name: _____ Date: _____

An observer traveling along a smooth 3D trajectory $\vec{r}(t)$ experiences an intrinsic, moving coordinate system $\{\hat{T}, \hat{N}, \hat{B}\}$ adapted entirely to their motion. This triad decomposes path behavior into pure geometry: forward direction, bending, and twisting.

1 Part 1: The Circular Helix (Constant Rates)

Consider a particle tracing the helical path:

$$\vec{r}(t) = \langle 40 \cos(t), 40 \sin(t), 9t \rangle, \quad t \geq 0$$

Problem 1: Velocity, Speed, and Heading ($\hat{T}$)

- (a) Compute the velocity vector $\vec{v}(t) = \vec{r}'(t)$ and the speed $v(t) = \|\vec{v}(t)\|$.
- (b) Explain why the speed $v(t)$ simplifies to a constant scalar.
- (c) Compute the **unit tangent vector** representing instantaneous heading:

$$\hat{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}$$

Problem 2: Turning and the Normal Direction ($\hat{N}$)

Because $\|\hat{T}(t)\| = 1$, any change $\hat{T}'(t)$ reflects purely a change in *direction*, never speed ($\hat{T} \cdot \hat{T}' = 0$).

- (a) Compute the derivative $\hat{T}'(t)$ and its magnitude $\|\hat{T}'(t)\|$.
- (b) Normalize $\hat{T}'(t)$ to find the **principal unit normal vector**:

$$\hat{N}(t) = \frac{\hat{T}'(t)}{\|\hat{T}'(t)\|}$$

- (c) Geometrically, which axis does $\hat{N}(t)$ point directly toward at all times?

Problem 3: Completing the Triad ($\hat{B}$)

To form a right-handed orthonormal coordinate system along the path, construct the **unit binormal vector**:

$$\hat{B}(t) = \hat{T}(t) \times \hat{N}(t)$$

- (a) Compute the cross product $\hat{B}(t)$.
- (b) Verify that $\|\hat{B}(t)\| = 1$ and check that $\hat{B}(t) \cdot \hat{T}(t) = 0$.

Problem 4: How the Curve Bends (Curvature κ)

The vector $\widehat{\mathbf{T}}'(t)$ measures how fast heading changes per unit *time*. To extract an intrinsic property of the path independent of traversal speed, we measure rate of change per unit **arc length** s ($ds = v dt$):

$$\frac{d\widehat{\mathbf{T}}}{ds} = \frac{\widehat{\mathbf{T}}'(t)}{\|\mathbf{r}'(t)\|}$$

The magnitude of this rate is the **curvature** κ :

$$\kappa = \left\| \frac{d\widehat{\mathbf{T}}}{ds} \right\| = \frac{\|\widehat{\mathbf{T}}'(t)\|}{\|\mathbf{r}'(t)\|}$$

- (a) Calculate κ for this helix.
- (b) A flat circle of radius $R = 40$ in the xy -plane has curvature $\kappa = \frac{1}{R} = \frac{1}{40} = \frac{40}{1600}$. Compare your value ($\frac{40}{1681}$) to $\frac{1}{40}$. Why does the upward pitch ($9t$) cause the helix to bend less sharply than the flat circle?

Problem 5: How the Curve Twists (Torsion τ)

The plane spanned by $\{\widehat{\mathbf{T}}, \widehat{\mathbf{N}}\}$ is the **osculating plane** (the plane of instantaneous turning). The binormal $\widehat{\mathbf{B}}$ is orthogonal to it. If the curve lies flat in a plane, $\widehat{\mathbf{B}}$ is constant. The rate at which the osculating plane tilts out of planarity is measured by the **torsion** τ :

$$\frac{d\widehat{\mathbf{B}}}{ds} = -\tau\widehat{\mathbf{N}} \quad \implies \quad \frac{\widehat{\mathbf{B}}'(t)}{\|\mathbf{r}'(t)\|} = -\tau\widehat{\mathbf{N}}(t)$$

- (a) Compute $\widehat{\mathbf{B}}'(t)$ and evaluate $\frac{d\widehat{\mathbf{B}}}{ds} = \frac{\widehat{\mathbf{B}}'(t)}{\|\mathbf{r}'(t)\|}$.
- (b) Show that $\frac{d\widehat{\mathbf{B}}}{ds}$ is indeed a scalar multiple of $\widehat{\mathbf{N}}(t)$, and extract the exact constant τ .

2 Part 2: A Closed 3D Loop (Variable Curvature and Torsion)

Consider a particle tracing the closed saddle loop lying along the cylinder $x^2 + y^2 = 1$:

$$\mathbf{r}(t) = \langle \cos(t), \sin(t), \sin(2t) \rangle, \quad t \in [0, 2\pi]$$

Problem 6: Geometry and Variable Speed

- (a) Verify that $\mathbf{r}(0) = \mathbf{r}(2\pi)$. What are the minimum and maximum z -coordinates reached by this loop?
- (b) Compute the velocity $\vec{v}(t) = \mathbf{r}'(t)$ and the acceleration $\vec{a}(t) = \mathbf{r}''(t)$.
- (c) Write an expression for the speed $v(t) = \|\mathbf{r}'(t)\|$. Is speed constant along this path?

Problem 7: Calculating at Key Benchmark Points

Because the general algebraic expressions for $\widehat{\mathbf{T}}(t)$ and $\widehat{\mathbf{N}}(t)$ involve variable square roots, we compute the geometric quantities κ and τ at two specific locations using the cross-product formulas:

$$\kappa = \frac{\|\vec{v} \times \vec{a}\|}{v^3}, \quad \tau = \frac{(\vec{v} \times \vec{a}) \cdot \mathbf{r}'''}{\|\vec{v} \times \vec{a}\|^2}$$

(a) **At the crossing point** $t = 0$:

- Evaluate $\vec{v}(0)$, $\vec{a}(0)$, and $\vec{r}'''(0)$.
- Compute $\vec{v}(0) \times \vec{a}(0)$ and find $\kappa(0)$ and $\tau(0)$.

(b) **At the crest** $t = \frac{\pi}{4}$:

- Evaluate $\vec{v}(\frac{\pi}{4})$, $\vec{a}(\frac{\pi}{4})$, and $\vec{r}'''(\frac{\pi}{4})$.
- Compute $\vec{v}(\frac{\pi}{4}) \times \vec{a}(\frac{\pi}{4})$ and find $\kappa(\frac{\pi}{4})$ and $\tau(\frac{\pi}{4})$.

(c) Compare the values: does this closed loop have constant curvature or constant torsion?

3 What We Learned: The Moving Frame in Matrix Form

Everything you just calculated fits into two elegant, unified matrices.

1. The Frenet–Serret Matrix (Rates of the Frame)

As you move along any space curve parameterized by arc length s , the rate at which your personal frame rotates is governed by a single **skew-symmetric matrix**:

$$\frac{d}{ds} \begin{bmatrix} \hat{T} \\ \hat{N} \\ \hat{B} \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{bmatrix} \begin{bmatrix} \hat{T} \\ \hat{N} \\ \hat{B} \end{bmatrix}$$

Notice the structure:

- $\hat{T}$ only turns toward $\hat{N}$ at rate κ .
- $\hat{B}$ only tilts along $\hat{N}$ at rate $-\tau$.
- $\hat{N}$ is pulled from both sides: bent back by the tangent ($-\kappa\hat{T}$) and twisted by the binormal ($+\tau\hat{B}$).

2. Kinematics Decomposed onto $\{\hat{T}, \hat{N}, \hat{B}\}$

If you take successive time derivatives of position $\vec{r}(t)$, the motion vectors project into your moving coordinate triad as a lower-triangular matrix:

$$\begin{bmatrix} \vec{v} \\ \vec{a} \\ \vec{j} \end{bmatrix} = \begin{bmatrix} \vec{r}' \\ \vec{r}'' \\ \vec{r}''' \end{bmatrix} = \begin{bmatrix} v & 0 & 0 \\ v' & \kappa v^2 & 0 \\ v'' - \kappa^2 v^3 & 3\kappa v v' + \kappa' v^2 & \kappa \tau v^3 \end{bmatrix} \begin{bmatrix} \hat{T} \\ \hat{N} \\ \hat{B} \end{bmatrix}$$

Takeaway:

- **Velocity** $\vec{v}$ lives entirely along $\hat{T}$.
- **Acceleration** $\vec{a}$ lives entirely in the osculating plane spanned by $\hat{T}$ and $\hat{N}$.
- **Jerk** $\vec{j} = \vec{r}'''$ is the very first kinematic derivative to break out of the plane into the binormal direction $\hat{B}$, with magnitude proportional to both curvature and torsion ($\kappa \tau v^3$).

4 Part 3: Independent Synthesis

Problem 8: The Tilted Path

Consider a particle traveling along the path:

$$\vec{r}(t) = \langle 3 \sin(t), 4 \sin(t), 5 \cos(t) \rangle$$

Calculate the torsion $\tau(t)$ for all t . Based entirely on your resulting value for τ , what does this tell you about the macroscopic geometry of this trajectory?

Problem 9: The Vanishing Tangent

A foundational requirement for a smooth curve is that it never stops to sharply change direction. Consider a particle whose position is given by the planar curve:

$$\vec{r}(t) = \langle t^2, t^3, 0 \rangle$$

- (a) Compute the velocity $\vec{v}(t)$ and evaluate it at $t = 0$.
- (b) Attempt to calculate the unit tangent vector $\hat{T}(0)$. What algebraic issue arises?
- (c) Sketch the path in the xy -plane (by noting that $y^2 = x^3$). What physical geometric feature exists at the origin that causes the moving frame to completely break down?

Problem 10: The Flipping Osculating Plane

Consider another planar path, $\vec{r}(t) = \langle t, t^3, 0 \rangle$. Unlike the previous problem, the velocity here never vanishes.

- (a) Compute the cross product $\vec{v}(t) \times \vec{a}(t)$.
- (b) The unit binormal vector $\hat{B}(t)$ always points in the same direction as $\vec{v} \times \vec{a}$. Determine the exact, constant vector $\hat{B}$ for $t > 0$, and the exact vector $\hat{B}$ for $t < 0$.
- (c) What happens to $\hat{B}(t)$ instantaneously at $t = 0$? (Hint: look at the curvature κ at $t = 0$). What geometric feature of a cubic graph causes the osculating plane to abruptly flip upside down?

Problem 11: Zero Curvature

Suppose a particle travels along a smooth path and a sensor records that its curvature is identically zero ($\kappa(s) = 0$ for all arc lengths s).

- (a) Using the first Frenet-Serret equation, $\frac{d\hat{T}}{ds} = \kappa\hat{N}$, what must be true about the derivative of the unit tangent vector?
- (b) If the rate of change of the unit tangent vector is zero, what does this tell you about the heading of the particle?
- (c) Based purely on this geometric deduction, what is the only possible macroscopic shape of this trajectory?

Problem 12: Geometric Deduction

A particle moves through space with a constant speed $v = 4$. Its moving frame is tracked, and it is found that the curvature is a constant $\kappa = \frac{1}{2}$ and the torsion is identically $\tau = 0$ for all time t .

Without computing any vectors, describe the exact geometric shape and size of this path. Justify your reasoning using the definitions of κ and τ .